Scale anomaly and quantum chaos in billiards with pointlike scatterers

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We argue that the random-matrix-like energy spectra found in billiards with pointlike scatterers are related to the quantum violation of scale invariance of a classical analog system. It is shown that the asymptotic freedom as expressed in the behavior of the running coupling constant explains the key characteristics of the level statistics of the system. [S1063-651X(96)12509-5]

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The concepts of scale anomaly and asymptotic freedom are among the key features of the gauge field theories which describe the interaction of the elementary particles. It is less widely recognized, however, that the scale anomaly can be found in a vastly simpler setting of one particle quantum mechanics. A particle scattered off a pointlike scatterer in two spatial dimension is known to have an energy dependent *s*-wave phase shift defying the scale invariance of its classical analog [1]. There exists a sister problem of particle motion confined in a hard-wall boundary with a pointlike scatterer inside. When the shape of the boundary is a rectangle, the problem belongs to a larger category of systems known as pseudointegrable billiards [2–5], a special subset of billiard problems [6,7]. This system is known for puzzling statistical properties of its energy eigenvalues [3,5]. It is shown through numerical experiments that the level statistics of the pseudointegrable billiard system resembles that of random-matrix ensembles [8], which are generally associated with chaotic dynamics [9]. This is in seeming contradiction to the absence of chaotic dynamics in a classical analog system. Further, when the levels are collected at a higher energy region, the level statistics moves toward the Poisson distribution which characterizes the integrable classical dynamics. Also, the system tends to show more Wigner-like statistics when the genus of the billiard system is increased; that is, in the present context, when the number of the singular scatterers is increased. These facts have never received sufficient explanation, in spite of several attempted studies based on the semiclassical periodic orbit quantization theory [10,11].

In this paper, we discuss the scale anomaly of billiards with pointlike scatterers as it is reflected in the spectral properties of the system. Specifically, we derive expressions for the energy and number-of-scatterer dependence of the effective coupling constant which predicts the level statistics that belong to the aforementioned "pseudointegrable class." We show the results of numerical calculations which corroborate this argument.

We consider a quantum particle of mass M moving freely inside a boundary B in two spatial dimensions on which its wave functions are assumed to vanish. We denote the eigenvalue and eigenfunction of the system as ε_n and ϕ_n, namely,

$$-\frac{1}{2M}\nabla^2\phi_n(\vec{x})=\varepsilon_n\phi_n(\vec{x}), \tag{1a}$$

with

$$\phi_n(\vec{x}_B)=0 \quad \text{where} \quad \vec{x}_B\in B. \tag{1b}$$

Assuming $\phi_n(\vec{x})$ to be normalized to unity, the Green's function is given by

$$G^{(0)}(\vec{x},\vec{x}\,';\omega)=\sum_{n=1}^{\infty}\frac{\phi_n(\vec{x})\,\phi_n(\vec{x}\,')}{\omega-\varepsilon_n}. \tag{2}$$

When the shape of the boundary B is such that the classical motion of the particle is regular (as in the case of B being rectangular), the quantum eigenvalues exhibit the so-called Poisson statistics. That is, the nearest neighbor spacing $s_n=\varepsilon_{n+1}-\varepsilon_n$ is distributed according to the Poisson distribution $P(s)=\exp(-s)$, and the spectral rigidity as the function of energy range L, takes the form $\Delta_3(L)=L/15$ [8].

We now place a pointlike scatterer at $\vec{x}_0$. Naively, one defines the scatterer in terms of the Dirac's δ function in two dimensions,

$$V(\vec{x})=v\,\delta(\vec{x}-\vec{x}_0). \tag{3}$$

Under the scale transformation $\vec{x}\rightarrow a\vec{x}$, the potential is transformed as $V(\vec{x})\rightarrow V(\vec{x})/a^2$. Since this behavior is identical to that of the Laplacian operator in Eq. (1a), the system is *scale invariant*. One expects, therefore, that the dynamical properties of the system should not depend on the energy. Formally, the transition matrix T (T matrix) in the presence of the scatterer V is given by

$$T=V+VG^{(0)}T. \tag{4}$$

The poles of T give the eigenvalues of the system. Because of the separability of the δ potential, $\langle\phi_n V\phi_m\rangle=v\,\phi_n(\vec{x}_0)\,\phi_m(\vec{x}_0)$, the T matrix is also separable:

$$\langle\phi_n T\phi_m\rangle=t(\omega)\,\phi_n(\vec{x}_0)\,\phi_m(\vec{x}_0). \tag{5}$$

Apart from the trivial solution $\omega=\varepsilon_n$ for the case of $\phi_n(\vec{x}_0)=0$, the poles of T are formally given by the roots of the equation

$$\frac{1}{t(\omega)}=\frac{1}{v}-G^{(0)}(\omega)=0, \tag{6}$$

with

$$G^{(0)}(\omega)\equiv G^{(0)}(\vec{x}_0,\vec{x}_0;\omega)=\sum_{n=1}^{\infty}\frac{\phi_n(\vec{x}_0)^2}{\omega-\varepsilon_n}. \tag{7}$$

However, Eq. (6), as it stands, is meaningless since

$$\sum_{n=1}^{\infty}\frac{\phi_n(\vec{x}_0)^2}{\omega-\varepsilon_n}\approx\langle\phi(\vec{x}_0)^2\rangle\sum_{n=1}^{\infty}\frac{1}{\omega-\varepsilon_n}$$
$$\approx\langle\phi(\vec{x}_0)^2\rangle\rho_0\int_0^{\infty}d\varepsilon\frac{1}{\omega-\varepsilon}\rightarrow\infty \tag{8}$$

where $\langle\phi(\vec{x}_0)^2\rangle$ is the average value of $\phi_n(\vec{x}_0)^2$ among various n. The divergence is brought about because the density of states is constant (which we denote ρ_0) with respect to the energy. To handle the divergence, a scheme for regularization and renormalization is called for. The most mathematically satisfying scheme is given by the self-adjoint extension theory of functional analysis [12,13]. Here we just quote the result. After the self-adjoint extension, the transition matrix $t(\omega)$ is given by

$$\frac{1}{t(\omega)}=\frac{(\omega-i\Lambda)}{1-e^{i\Theta}}\int d\vec{x}G^{(0)}(\vec{x},\vec{x}_0;\omega)G^{(0)}(\vec{x},\vec{x}_0;i\Lambda)$$
$$+\frac{(\omega+i\Lambda)}{1-e^{-i\Theta}}\int d\vec{x}G^{(0)}(\vec{x},\vec{x}_0;\omega)G^{(0)}(\vec{x},\vec{x}_0;-i\Lambda). \tag{9}$$

Here Λ is an arbitrary scale of the regularization, and Θ ($0\leqslant\Theta<2\pi$) is the parameter of self-adjoint extension. With the straightforward calculation, we find that the energy eigenvalues of the system — the poles of $t(\omega)$ — are determined by the equation

$$\frac{1}{\overline{v}}-\overline{G(\omega)}=0, \tag{10}$$

where

$$\overline{G(\omega)}=\sum_{n=1}^{\infty}\phi_n(\vec{x}_0)^2\left[\frac{1}{\omega-\varepsilon_n}+\frac{\varepsilon_n}{\varepsilon_n^2+\Lambda^2}\right] \tag{11}$$

is the regularized version of $G^{(0)}(\omega)$, and

$$\overline{v}=\left[\frac{\Lambda\sin\Theta}{1-\cos\Theta}\sum_{n=1}^{\infty}\frac{\phi_n(\vec{x}_0)^2}{\varepsilon_n^2+\Lambda^2}\right]^{-1} \tag{12}$$

is the formal (or renormalized) coupling strength of the scatterer. We stress that in spite of the purely mathematical construction of Eqs. (10)–(12), it does correspond to the physical small-size limit of the problem of a finite-size obstacle [14].

Since the series of Eq. (11) is convergent, the problem is now well defined. We look at the behavior of Eq. (10) at the high energy region $\omega\gg\Lambda$. For a given value of ω, we can approximate Eq. (11) by truncating the summation at $n=n_x(\omega)$,

$$\overline{G(\omega)}\approx\sum_{n=1}^{n_x(\omega)}\phi_n(\vec{x}_0)^2\left[\frac{1}{\omega-\varepsilon_n}+\frac{\varepsilon_n}{\varepsilon_n^2+\Lambda^2}\right] \tag{13}$$

with an error given by

$$\delta\overline{G}=\sum_{n=n_x(\omega)+1}^{\infty}\phi_n(\vec{x}_0)^2\left[\frac{1}{\omega-\varepsilon_n}+\frac{\varepsilon_n}{\varepsilon_n^2+\Lambda^2}\right]$$
$$\approx\langle\phi(\vec{x}_0)^2\rangle\rho_0\int_{\varepsilon_x(\omega)}^{\infty}d\varepsilon\left[\frac{1}{\omega-\varepsilon}+\frac{\varepsilon}{\varepsilon^2+\Lambda^2}\right]$$
$$\approx-\langle\phi(\vec{x}_0)^2\rangle\rho_0^2\frac{\omega}{n_x(\omega)}, \tag{14}$$

where we have used $\varepsilon_x(\omega)=n_x(\omega)/\rho_0$. Therefore, we can set

$$n_x(\omega)=\alpha\omega, \tag{15}$$

where α is a constant inversely proportional to the allowable error $\delta\overline{G}$. Once the summation is truncated at finite terms, we can rewrite Eq. (10) as

$$\frac{1}{v_{\rm eff}(\omega,\overline{v})}-\sum_{n=1}^{n_x(\omega)}\frac{\phi_n(\vec{x}_0)^2}{\omega-\varepsilon_n}=0, \tag{16}$$

with the effective strength $v_{\rm eff}(\omega,\overline{v})$ defined through

$$\frac{1}{v_{\rm eff}(\omega,\overline{v})}=\frac{1}{\overline{v}}-\sum_{n=1}^{n_x(\omega)}\phi_n(\vec{x}_0)^2\frac{\varepsilon_n}{\varepsilon_n^2+\Lambda^2}. \tag{17}$$

Comparing Eqs. (16) and (6), one realizes that the problem is now turned into an eigenvalue problem with finite basis states ϕ_n, $n=1,\dots,n_x(\omega)$. Although the system originally has no inherent scale, the effective coupling strength $v_{\rm eff}$ has an energy scale Λ, and, as a result, it acquires energy dependence. This is possibly the simplest example of the scale anomaly [1]. The effective coupling $v_{\rm eff}$ is also referred to as *running coupling* because of its energy dependence. When $v_{\rm eff}$ is large for the energy region of the interest, it induces the mixing among the basis states, and results in the so-called Wigner level statistics. That is, the nearest neighbor spacing distribution is given by $P(s)=1/2\,\pi s\exp(-1/4\,\pi s^2)$, and the spectral rigidity $\Delta_3(L)$ becomes considerably more rigid and approximately given by $\Delta_3(L)\approx 1/\pi^2\ln L-0.006\,95$ for $L\gg 1$ [8]. It is known that $v_{\rm eff}$ is large only in one energy region determined by the value of $\overline{v}$ and Λ [5]. Replacing the summation in Eq. (17) with the integral in Eq. (14), we have

$$v_{\rm eff}(\omega,\overline{v})\approx\frac{\overline{v}}{1-\overline{v}\langle\phi(\vec{x}_0)^2\rangle\rho_0\ln\sqrt{1+(n_x(\omega)/\rho_0\Lambda)^2}}. \tag{18}$$

At the limit $\omega\rightarrow\infty$, we have $\ln\sqrt{1+(n_x(\omega)/\rho_0\Lambda)^2}\approx\ln n_x(\omega)\approx\ln\omega$. We arrive at

$$v_{\rm eff}(\omega,\overline{v}) \approx -\frac{1}{\langle \phi(\vec{x}_0)^2\rangle \rho_0 \ln\omega} \quad (\omega\to\infty). \tag{19}$$

Remarkably, the formal strength disappears from the expression of the effective strength $v_{\rm eff}$ in the high energy limit. Therefore, the level statistics of the billiard system with a pointlike scatterer is predicted to become more Poisson-like at higher energy region, irrespective of the choice of the formal coupling $\overline{v}$. The fact that the strength $v_{\rm eff}$ disappears at the limit $\omega\to\infty$ goes along well with our intuition that, at the classical limit, a pointlike obstacle has no effect on the motion of a particle. At this limit, we have *logarithmic asymptotic freedom* where all the wave functions are unperturbed, and the scale invariance is restored in a trivial manner. We note that when we replace the sum with the integral in obtaining Eq. (18), we implicitly assume that the size of the billiard is far larger than the scale in discussion. That is the reason why the size of the billiard boundary, which obviously breaks the scale invariance, does not appear in our arguments.

We next consider the case of two pointlike scatterers. We place the scatterers at $\vec{x}_0$ and $\vec{x}_1$, with formal strengths $\overline{v_0}$ and $\overline{v_1}$, respectively. The eigenvalues of this system are determined by

$$\begin{vmatrix} \dfrac{1}{\overline{v_0}} - \overline{G_{00}}(\omega) & -G_{01}^{(0)}(\omega) \\ -G_{10}^{(0)}(\omega) & \dfrac{1}{\overline{v_1}} - \overline{G_{11}}(\omega) \end{vmatrix} = 0, \tag{20}$$

where $\overline{G_{ij}}(\omega)$ and $G_{ij}^{(0)}(\omega)$ are defined as

$$\overline{G_{ij}}(\omega) = \sum_{n=1}^{\infty} \phi_n(\vec{x}_i)\phi_n(\vec{x}_j)\left[\frac{1}{\omega-\varepsilon_n} + \frac{\varepsilon_n}{\varepsilon_n^2+\Lambda^2}\right] \tag{21}$$

and

$$G_{ij}^{(0)}(\omega) = \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}_i)\phi_n(\vec{x}_j)}{\omega-\varepsilon_n}. \tag{22}$$

Let us suppose that two scatterers are placed closely to each other. We truncate the sum of $\overline{G_{00}}(\omega)$ and $\overline{G_{11}}(\omega)$ at $n=n_x(\omega)$, as before. As for $G_{01}^{(0)}(\omega)$, which is finite as long as $\vec{x}_0 \neq \vec{x}_1$, the same truncation is possible if two scatterers are apart by

$$|\vec{x}_1-\vec{x}_0| \simeq \frac{1}{\sqrt{\varepsilon_x}} = \left(\frac{\rho_0}{n_x(\omega)}\right)^{1/2}, \tag{23}$$

since the contributions from higher n, which probe finer length scale than $|\vec{x}_1-\vec{x}_0|$, cancel among themselves. Below $n \leq n_x(\omega)$, we can approximate $\phi_n(\vec{x}_0) \simeq \phi_n(\vec{x}_1)$, since $\phi_n(\vec{x})$ is slowly oscillating in the distance $|\vec{x}_1-\vec{x}_0|$. The matrix equation, Eq. (20), then can be reduced to

$$\frac{1}{v_{\rm eff}^{(2)}(\omega,\overline{v_0},\overline{v_1})} - \sum_{n=1}^{n_x(\omega)} \frac{\phi_n(\vec{x}_0)^2}{\omega-\varepsilon_n} = 0 \tag{24}$$

with the effective coupling $v_{\rm eff}^{(2)}(\omega,\overline{v_0},\overline{v_1})$ given by

$$v_{\rm eff}^{(2)}(\omega,\overline{v_0},\overline{v_1}) = v_{\rm eff}(\omega,\overline{v_0}) + v_{\rm eff}(\omega,\overline{v_1}). \tag{25}$$

This equation reveals an interesting feature of the system with two pointlike scatterers. If $\overline{v_0}$ differs from $\overline{v_1}$ appreciably, $v_{\rm eff}(\omega,\overline{v_0})$ and $v_{\rm eff}(\omega,\overline{v_1})$ become large at different energies. This means that the particle moving in the billiard system *cannot see* the two scatterers at the same time for any given energy. On the other hand, the two scatterers disturb the particle in a coherent manner when $\overline{v_0} \simeq \overline{v_1}$. With Eq. (19), in the limit of $\overline{v_1} = \overline{v_0}$ we obtain

$$\begin{aligned} v_{\rm eff}^{(2)}(\omega,\overline{v_0},\overline{v_0}) &= 2v_{\rm eff}(\omega,\overline{v_0}) \\ &\approx -\frac{2}{\langle \phi(x_0)^2\rangle \rho_0 \ln\omega} \quad (\omega\to\infty). \end{aligned} \tag{26}$$

That is, two closely placed pointlike scatterers with the same formal strength act as a single scatterer of twice the *effective* strength.

If we remove our assumption of two pointlike scatterers located closely together, we have to deal with Eq. (20) directly. However, we do not expect an essential change of the character of level statistics, since the statistical measures are known to be rather insensitive to the precise location of the scatterers. This allows us to generalize our findings to the case of more than two scatterers. Let us consider a billiard system with an arbitrary (finite) number of pointlike scatterers inside. We classify the scatterers according to the magnitude of the formal strength; we collect the scatterers with the same order of magnitude of the formal strength as a single group. We can expect that the scatterers belonging to one of such groups disturb the particle motion in a coherent manner in the energy region determined by their own formal strength, while their influence never appears at different energies. In particular, in the case of scatterers with a common formal strength, their effects are additive. Therefore, it is predicted that one should obtain more Wigner-like level statistics by increasing the number of singular scatterers.

We now show the numerical data obtained from a rectangular billiard system with several pointlike scatterers that demonstrate the predicted behavior of the level statistics. In Fig. 1, we display the spectral rigidity $\Delta_3(L)$ of a rectangular billiard system of size $[L\times 1/L]$, $L=\pi/e\approx 1.155\,72$ with a singular scatterer of strength $\overline{v}=0.15$ placed at $\vec{x}_0 = (0.6180L, 0.414\,21/L)$. The mass of particle is set to $M=2\pi$ in units of mass scale $\Lambda=1$. This choice of parameters makes the effective coupling strongest at the region around zero energy [5]. The statistics are taken separately from 200th – 3200th, 3200th – 6200th, 6200th – 9200th, and 9200th – 12200th states as indicated in the caption. In the calculation, the unperturbed basis states with energy of up to five times the energy when consideration are included. The movement toward Poisson statistics at higher energy, as predicted by Eq. (18), is clearly visible. In Fig. 2, the rigidity with one, two, and three scatterers are shown. The trend toward more Wigner-like statistics for more scatterers is observed as is expected from Eq. (26). We note that the nearest neighbor spacing $P(s)$ also shows a corresponding behavior.

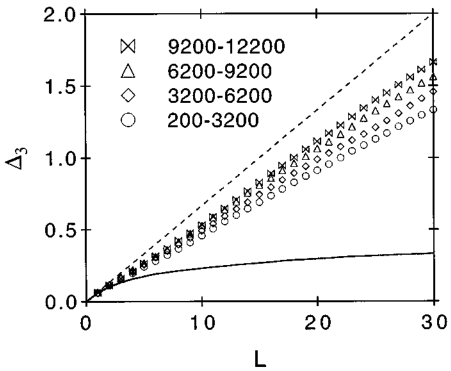

FIG. 1. Spectral rigidity at various energy regions of a rectangular billiard of size $[L\times 1/L]$ $(L=1.155\,72)$ with a singular scatterer placed at $\vec{x}_0=(0.618\,03L, 0.414\,21/L)$. The parameters are $M=2\pi$, $\Lambda=1$, and $\overline{v}=0.15$. The solid and dashed lines are the predictions of Wigner and Poisson statistics.

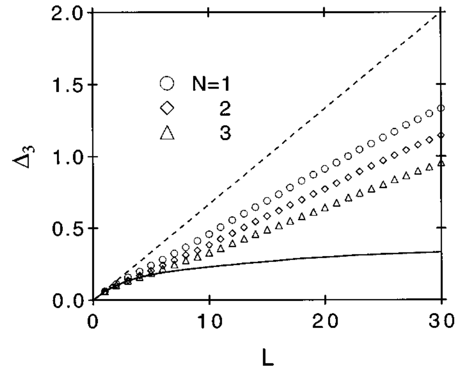

FIG. 2. Spectral rigidity calculated from 200th to 3200th states with one [at $(0.618\,03L, 0.414\,21/L)$], two [second one at $(0.30277L, 0.23606/L)$], and three [third one at $(0.807\,41L, 0.837\,72/L)$] singular scatterers. $M=2\pi$, $\Lambda=1$, and $\overline{v}=0.15$ for all scatterers.

Finally, we discuss our results in a broader context. The standard approach in quantum chaology has been the semiclassical periodic orbit theory [15,16] which is a truncation of the WKB approximation. Recently, there have been successful extensions of the periodic orbit theory to include ''diffractive orbits'' as the leading quantum correction [17,18]. These approaches appear promising in treating a more generic pseudointegrable billiard system with finite-size objects with sharp edges (the type discussed in Refs. [2,3,11]). However, they may not be easily applied to the present problem, since the semiclassical treatment of the diffractive effects around the pointlike scatterer is currently not well established. Our approach, on the other hand, is fully quantum mechanical without any resort to semiclassical approximations. Although no direct link can be established at this point, our results certainly give an insight into the problem of quantum level statistics of more generic pseudointegrable systems.

The spectral statistics of the billiard system studied here is clearly a pedagogical manifestation of the quantum violation of classical scale invariance. We believe that it enhances the intuitive understanding of the scale invariance and asymptotic freedom which have been viewed as phenomena found only in the esoteric theories of elementary particles.

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